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LETTER TO THE EDITOR

Critical behaviour in aperiodic systems

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Abstract. We consider layered systems with aperiodically modulated couplings and study the effect of inhomogeneity on the critical behaviour. Using scaling arguments a relevance/irrelevance-type criterion is formulated and for relevant inhomogeneities the scaling form of singular quantities is determined. We have performed exact calculations for the surface magnetization of aperiodic quantum Ising chains and for a $(1+1)$-dimensional directed polymer in a longitudinal aperiodic environment. The results for both systems are in accord with scaling considerations. For a marginal aperiodicity the critical exponents are found to be non-universal.

The discovery of quasicrystals has started a growing theoretical activity to understand their structure and physical properties (for a recent review see [1]). Among others an interesting theoretical question is how the imperfection of the underlaying lattice influences the critical properties of a system. Till now exact results of this problem have been restricted to models with an essentially one-dimensional aperiodicity—in higher dimensions the aviailable results are numerical. For example there is numerical evidence that phase transitions on two-dimensional Penrose quasilattices, such as the Ising model [2–4], percolation [5] and self-avoiding walks [6] are universal, i.e. the effect of the aperiodicity of this lattice is *irrelevant*.

Similar conclusions have been obtained by exact calculations on the quantummechanical phase transition of the one-dimensional transverse Ising model (TIM) on Fibonacci and related lattices [7–11]. The phase transition of the same system on some hierarchical and aperiodic lattices, however, was washed out by the inhomogeneity of the lattice; thus in these cases the effect of the aperiodicity is *relevant* [12, 13]. There are also some examples on *marginal* aperiodicities, where—according to numerical results [14]—the critical exponents are continuous functions of the strength of the inhomogeneity.

For the TIM on an aperiodic lattice Luck has pointed out [14] the relation between the size of the fluctuations in the couplings and the type of the phase transition in the system. For a sequence with *bounded fluctuations* the system undergoes an Ising-type phase transition like the homogeneous one, while for *unbounded fluctuations* this type of phase transition is absent.

In the following we study systems with a one-dimensional aperiodicity, such as the TIM or layered two- or three-dimensional models. The different units of the aperiodic lattice is denoted by the letters of the alphabet $(A, B, \ldots)$ and substitutional rules are used to generate the self-similar lattice. For example the Fibonacci sequence is built from two letters A and B according to the substitution $\sigma(A) = AB$ and $\sigma(B) = A$.

Basic properties of a sequence can be deduced from the substitutional matrix, whose columns contain the number of letters $A, B, \ldots$ in $\sigma(A), \sigma(B), \ldots$, respectively. For the Fibonacci lattice this matrix reads as

$$\underline{M} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

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The length of a word after n substitutions—L_n—can be expressed for large n with the largest eigenvalue of $\underline{M}$ as

$$L_n \simeq \Lambda_1^n \tag{2}$$

while fluctuations in the total number of individual letters $L_n^A, L_n^B, \ldots$ are connected with the next to leading eigenvalue. For example in two-letter substitutions

$$L_n^A - \frac{\rho_A}{\rho_B} L_n^B \simeq \Lambda_2^n . \tag{3}$$

Here ρ_A and ρ_B denotes the number densities of the different letters as $n \to \infty$—these are the components of the leading right eigenvector of the substitutional matrix. For the Fibonacci sequence $\Lambda_1 = \tau = (1+\sqrt{5})/2$ the golden mean and $\Lambda_2 = -1/\tau$. Since $|\lambda_2| < 1$ the fluctuations are bounded and the sequence is said to have the Pisot–Vijarayaghavan property [15].

A physical system on such a lattice (cf the Ising model) is assumed to have different couplings in the aperiodic direction, say $K_A, K_B, \ldots$ in accordance with the letters of the underlaying lattice. These couplings can be expressed with their average $\overline{K}$ and the strength of the inhomogeneity $\delta > 0$. For the two-letter substitution

$$K_A = \overline{K} + \rho_B \delta \qquad K_B = \overline{K} - \rho_A \delta . \tag{4}$$

Now let us consider a system which has a finite width l in the aperiodic direction, but infinite extent in the other ones, and calculate the fluctuation in the aperiodic couplings. According to general rigorous results [16] (see also [14])

$$\sum_{i=1}^{l} (K_i - \overline{K}) \approx \delta l^{\beta} F\left(\frac{\ln l}{\ln \lambda_1}\right) \tag{5}$$

where the β 'wandering exponent' can be expressed using (2) and (3) as

$$\beta = \frac{\ln |\Lambda_2|}{\ln \Lambda_1} \tag{6}$$

and $F(x)$ is a 'fractal function', which is periodic, $F(x) = F(x+1)$, continuous but nowhere differentiable†. If the system is at its bulk critical point $\overline{K} = K_c$ and the inhomogeneity represents a small perturbation $\delta \ll 1$, then for the finite strip local couplings differ on average from the critical one by an amount

$$\Delta K(l) \propto \delta l^{\beta - 1} . \tag{7}$$

Since according to (6) $\beta < 1$, $\Delta K(l)$ goes to zero in the thermodynamic limit, ΔK being an energy-like parameter that specifies the deviation from criticality transforms as

$$\Delta K'(l') = \Delta K(l) b^{1/\nu} \tag{8}$$

under a scaling transformation $\underline{r}' = \underline{r}/b$, where ν is the correlation length critical exponent. Comparing (7) with (8) one obtains the transformation law for the inhomogeneity parameter

$$\delta' = \delta b^{\Phi/\nu} \tag{9}$$

† Strictly speaking the wandering exponent should be non-negative, since the sum in (5) fluctuates on a scale of O(1). However, restricting the size of the system to L_ns, i.e. to use only complete words, one can also attribute to β negative numbers.

where the cross-over exponent

$$\Phi = 1 + \nu(\beta - 1)\,. \tag{10}$$

Thus the effect of aperiodicity is relevant, marginal and irrelevant for $\beta > 1 - 1/\nu$, $\beta = 1 - 1/\nu$ and $\beta < 1 - 1/\nu$, respectively.

This condition is consistent with existing exact results on aperiodic systems. For the TIM, where $\nu = 1$ the relevance/irrelevance criterion is related to the unbounded/bounded nature of the fluctuations, as observed in [14]. A further example is an interface in an aperiodic potential in the transverse direction. Under thermal fluctuations in $d \leqslant 3$ dimensions $\nu = \nu_\perp = (d-3)/2$ [17], therefore in $d = 3$—where $\nu_\perp = 0$—the aperiodicity is always relevant, unless $\Lambda_2 = 0$. This behaviour has been observed by RG studies [18]. We note that in three dimensions the marginality condition is satisfied for the Thue–Morse sequence $\sigma(A) = AB$ and $\sigma(B) = BA$, which therefore needs some specific treatment. On the other hand in two dimensions—according to (10)—the marginal value of the wandering exponent for an interface is $\beta = -1$, which is characteristic for the Fibonacci lattice, where non-universal critical behaviour was indeed found by exact calculations [19].

In the following—using scaling considerations—we show that for relevant inhomogeneities one can determine the scaling form of the singular quantities near the critical point. We start to write the transformation law for the magnetization in terms of the reduced temperature $t = (T - T_c)/T_c$ and the strength of the inhomogeneity δ as

$$m(t, \delta) = b^{-x} m(tb^{1/\nu}, \delta b^{\Phi/\nu}) \tag{11}$$

where x denotes the anomalous dimension of the magnetization. Similar scaling relations containing a relevant parameter like δ here, have already been analysed in context of smoothly inhomogeneous systems and parabolic geometries (for a review see [20]). In these problems results obtained by scaling considerations are in agreement with exact calculations on specific systems. In the following we make use of this analogy to determine the scaling behaviour of the thermodynamic quantities in systems with a relevant aperiodicity.

The first consequence of the presence of a relevant parameter in (11) is the existence of a new characteristic length

$$l = \delta^{-\nu/\Phi} \tag{12}$$

which stays finite at the critical point. Consequently critical correlations decay faster than a simple power law for $\delta > 0$. Repeating an argument of Burkhardt [21] (see also [20]) one obtains a stretched exponential behaviour

$$G(t = 0, \delta, r) \propto r^{-2x} \exp\left[-A(r/l)^{\Phi}\right] \tag{13}$$

where A is a positive constant. The size of a correlated domain in homogeneous systems is measured by the correlation length $\xi_h \sim |t|^{-\nu}$, while in systems with a relevant inhomogeneity this size can be estimated by equating the thermal $\sim t$ and the inhomogeneity energy $\sim \delta \xi_{in}^{(\Phi-1)/\nu}$ contributions:

$$\xi_{in} \sim \left(\frac{t}{\delta}\right)^{\nu/(\Phi-1)}\,. \tag{14}$$

Note that $\xi_{in} < \xi_h$ for $\Phi > 0$, i.e. for a relevant perturbation. Integrating the correlation function in (13) within the correlated domain of size ξ_{in} one obtains the singular behaviour of the susceptibility of the system:

$$\chi \sim \exp(-\text{const}\ \xi_{in}/\xi_h)\,. \tag{15}$$

The essential singularity in (15) should also be characteristic for other singular physical quantities. For example the singular part of the free energy behaves like

$$f_{\text{sing}} \sim \exp\left(-\text{const } t^{1/(\beta-1)+\nu}\delta^{1/(1-\beta)}\right) \tag{16}$$

and similar dependence on t and δ is expected for the spontaneous magnetization for $t < 0$.

At this point let us remark on the similarity of the structure of the scaling form of the magnetization in (11) and that for the local magnetizations in inhomogeneous systems [22, 20] and in parabolic geometries [23]. For an inhomogeneous system the characteristic length—like l in (12)—is connected with the strength of the perturbation, whereas for a system with a parabolic shape it is the parameter of the parabola. In these cases the stretched exponential decay of correlations—as in (13)—is connected with the existence of a smoothly varying local correlation length, which can be expressed by the local value of the coupling or corresponds to the local width of the parabola [20]. Exact results obtained on specific models with smooth inhomogeneities or in a parabolic geometry are in agreement with the scaling relations in (13), (15) and (16).

For systems with a relevant aperiodicity the only exact result is known about the ground-state energy singularity of the TIM [14], which is consistent with the corresponding formula in (16), taking $\nu = 1$. In the following we investigate a two-dimensional directed polymer system in the presence of a longitudinal aperiodic potential in order to check results obtained above by scaling considerations.

In the directed polymer model the polymer connects next-nearest-neighbour lattice sites of a square lattice in such a way that steps towards the negative x-axis are forbidden (fully directed self-avoiding walk) [24]. Due to the layered structure of the model the statistical weight of each step of the 2^l l-step walk is the same, thus the connectivity constant is $C = 2$. The monomer fugacity at step i follows the aperiodicity of the underlying lattice and can be written as

$$\omega_i = C^{-1}\exp(-t + u_i) \tag{17}$$

where $t > 0$ controls the deviation from criticality and plays the role of the reduced temperature, while u_i stands for the aperiodic external potential with zero average:

$$\lim_{l\to\infty}\frac{1}{l}\sum_{i=1}^{l} u_i = 0.$$

The grand partition sum of the problem reads as

$$\Xi - 1 = \sum_{l=1}^{\infty} C^l \prod_{i=1}^{l}\omega_i = \sum_{l=1}^{\infty}\exp(-tl)\exp\left(\sum_{i=1}^{l} u_i\right). \tag{18}$$

As t approaches zero Ξ develops a singularity, which in homogeneous systems behaves like $\Xi(t) \sim t^{-\gamma}$, while on a finite system of size L at the critical point $\Xi_L(0) \sim L^{\gamma}$. The susceptibility exponent γ in a homogeneous system is $\gamma_h = 1$, but the critical behaviour may be changed by aperiodic potentials. In this case the *canonical* partition sum scales according to (5) as

$$Z_l = \exp\left(\sum_{i=1}^{l} u_i\right) \sim \exp\left[\delta l^{\beta} F\left(\frac{\ln l}{\ln\lambda_1}\right)\right] \tag{19}$$

and its behaviour for large l is different for $\beta < 0$ and $\beta > 0$, respectively. For bounded fluctuations—$\beta < 0$—Z_l tends to unity, the susceptibility exponent $\gamma = \gamma_h = 1$, thus the perturbation is irrelevant in accordance with the relevance/irrelevance criterion in (10) with $\nu = \nu_\parallel = 1$.

On the other hand for unbounded fluctuations—$\beta > 0$—Z_l has an exponential dependence on l^β and the grand partition sum on finite systems scales as

$$\Xi_L(0) \sim \exp(\text{const}\ \delta L^\beta) \tag{20}$$

i.e. the L-dependence is of stretched exponential. To determine the t-dependence of Ξ first we estimate the average size of the polymer at $t > 0$ by equating the two exponential contributions to (18) with the result $\overline{L} \propto (\delta/t)^{1/1-\beta}$. Then setting $\overline{L}$ into (20) one arrives at

$$\Xi(t) \sim \exp\left(\text{const}\ t^{\beta/(\beta-1)}\delta^{1/(1-\beta)}\right) \tag{21}$$

with a result in accordance with the scaling form of the susceptibility in (15). The diverging behaviour of $\Xi(t)$, i.e. the positive sign of the arguments of the exponential functions in (20) and (21), is explained by the directed nature of the model. Due to this fact the polymer cannot become localized in an external potential, rather its susceptibility is grown in an anomalous way. We note that similar behaviour is observed when the directed polymer is placed into a relevant, smoothly inhomogeneous longitudinal potential [25].

Most interesting is the behaviour of the polymer for $\beta = 0$, i.e. in the marginal case, when the growth of fluctuations in (5) takes place on a logarithmic scale. According to rigorous results [16] the partial summations behave like

$$\frac{1}{L}\sum_{l=1}^{L}\left(\sum_{i=1}^{l} u_i\right)^{2k} = (2k-1)!!\ (w\delta^2 \ln L)^k + \mathrm{O}[(\ln L)^{k-1}] \tag{22}$$

where the w positive constant is characteristic for the substitution. Since the sums of odd powers in (22) are subleading by at least one power of $\ln L$ [16] in leading order the Taylor series of $\Xi_L(0)$ contains only the even powers:

$$\Xi_L(0) \approx \sum_{l=1}^{L}\sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(\sum_{i=1}^{l} u_i\right)^{2k} \tag{23}$$

which can be summed using (22) as

$$\Xi_L(0) \approx L\exp\left(\ln L\frac{w\delta^2}{2}\right) = L^{1+w\delta^2/2}. \tag{24}$$

Thus the critical behaviour is non-universal, and the susceptibility exponent is a continuous function of the strength of the inhomogeneity:

$$\gamma(\delta) = 1 + \frac{w}{2}\delta^2 + \mathrm{O}(\delta^4). \tag{25}$$

For the specific substitution $\sigma(A) = AB$, $\sigma(B) = AA$ with $\lambda_1 = 2$ and $\lambda_2 = -1$, thus $\beta = 0$, we have determined the behaviour of the susceptibility exactly. Considering the lattice after an even number of substitutions, i.e. when $L = 2^{2n}$, and analysing the structure of the sequence one can show that

$$\Xi_{2^{2n}}(0) = \left[2\mathrm{ch}\frac{\delta}{2}\right]^{2n} + \mathrm{e}^{\delta}. \tag{26}$$

Thus the susceptibility exponent is given by

$$\gamma(\delta) = 1 + \frac{\ln[\mathrm{ch}(\delta/2)]}{\ln 2}. \tag{27}$$

In the limiting cases $\gamma(\delta)$ can be expressed as

$$\gamma(\delta) = 1 + \frac{1}{8\ln 2}\delta^2 + \cdots \qquad \delta \ll 1 \tag{27a}$$

and

$$\gamma(\delta) = \frac{\delta}{2\ln 2} + \cdots \qquad \delta \gg 1. \tag{27b}$$

Equation (27*a*) is in accord with the general result in (25) with $w = 1/4\ln 2$; furthermore the $\delta \to \infty$ limit in (27*b*) joins the relevant perturbation result in (20) when $\beta \to 0^+$.

As another example we consider the surface magnetization of an aperiodic quantum Ising chain described by the Hamiltonian

$$H = -\tfrac{1}{2}\sum_{i=1}^{\infty}\left[K_i\sigma_i^x\sigma_{i+1}^x + \sigma_i^z\right] \tag{28}$$

where the σ_i^x, σ_i^z are Pauli matrices at site i, and the coupling

$$K_i = \exp[(t - u_i)/2] \tag{29}$$

follows an aperiodic sequence. According to a recent study of the problem by Turban and Berche [26] the critical point of the system is at $t = 0$ (in the ordered phase $t > 0$), if $\lim_{l\to\infty}(1/l)\sum_{i=1}^{l} u_i = 0$, in analogy with the previously studied directed polymer problem. This analogy goes even further considering the surface magnetization of the Ising model, which is given by the formula [26,27]

$$m^{\mathrm{s}} = \left[1 + \sum_{l=1}^{\infty}\prod_{i=1}^{l} K_i^{-2}\right]^{-1/2} = \left[1 + \sum_{l=1}^{\infty}\exp(-tl)\exp\left(\sum_{i=1}^{l} u_i\right)\right]^{-1/2} \tag{30}$$

and its asymptotic behaviour $m^{\mathrm{s}}(t) \propto t^{\beta_{\mathrm{s}}}$ and $m_L^{\mathrm{s}}(0) \propto L^{-\beta_{\mathrm{s}}}$ is governed by the surface magnetization exponent β_{s}.

Comparing (18) with (30) one can notice the correspondences

$$m_L^{\mathrm{s}}(t) \Longleftrightarrow [\Xi_L(t)]^{-1/2} \tag{31a}$$

and

$$\beta_{\mathrm{s}} \Longleftrightarrow \gamma/2. \tag{31b}$$

Thus one can easily see that for bounded fluctuations ($\beta < 0$) the surface magnetization exponent $\beta_{\mathrm{s}} = \frac{1}{2}$ is the same as in a homogeneous system. This observation is in accord with our relevance/irrelevance criterion in (10) and with explicit results on specific sequences in [26]. For unbounded fluctuations ($\beta > 0$) according to (20), (21) and (31*a*) the decay of the surface magnetization is anomalous. Finally, in the marginal case ($\beta = 0$) the surface magnetization exponent is non-universal, and depends on the strength of the inhomogeneity as given in (25), (27) and (31*b*).

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References

[1] Steinhardt P and DiVincenzo D (eds) 1991 *Quasicrystals: The State of the Art* (Singapore: World Scientific)
[2] Godrèche C, Luck J M and Orland H 1986 *J. Stat. Phys.* **45** 777
[3] Okabe Y and Niizeki K 1988 *J. Phys. Soc. Japan* **57** 1536
[4] Sørensen E S, Jarić M V and Ronchetti M 1991 *Phys. Rev.* B **44** 9271
[5] Sakamoto S, Yonezawa F, Aoki K, Nosé S and Hori M 1989 *J. Phys. A: Math. Gen.* **22** L705
[6] Langie G and Iglói F 1992 *J. Phys. A: Math. Gen.* **25** L487
[7] Iglói F 1988 *J. Phys. A: Math. Gen.* **21** L911
[8] Doria M M and Satija I I 1988 *Phys. Rev. Lett.* **60** 444
[9] Benza G V 1989 *Europhys. Lett.* **8** 321
[10] Henkel M and Patkós A 1992 *J. Phys. A: Math. Gen.* **25** 5223
[11] Lin Z and Tao R 1992 *J. Phys. A: Math. Gen.* **25** 2483
[12] Tracy C A 1988 *J. Phys. A: Math. Gen.* **21** L603
[13] Cecatto H A 1989 *Z. Phys.* B **75** 253
[14] Luck J M 1992 *Saclay preprint* SPhT/92/136
[15] Pisot C 1938 *Ann. Scuola Norm. Sup. Pisa (Italy)* **7** 205
[16] Dumont J M 1990 *Number Theory and Physics (Springer Proceedings in Physics 47)* ed J M Luck, P Moussa and M Waldschmidt (Berlin: Springer)
[17] Fisher M E 1984 *J. Stat. Phys.* **34** 667
[18] Lipowsky R and Henley C L 1988 *Phys. Rev. Lett.* **60** 2394
[19] Henley C L and Lipowsky R 1987 *Phys. Rev. Lett.* **59** 1679
[20] Iglói F, Peschel I and Turban L 1993 *Preprint*
[21] Burkhardt T W 1982 *Phys. Rev. Lett.* **48** 216
[22] Privman V and Švrakić N M 1989 Directed models of polymers, interfaces and finite-size properties *Lecture Notes in Physics 338* (Berlin: Springer)
[23] Iglói F 1992 *Europhys. Lett.* **19** 305
[24] Turban L and Berche B 1993 *Nancy preprint*
[25] Peschel I 1984 *Phys. Rev.* B **30** 6783